LOCALLY SOLID TOPOLOGIES

UNDERGRADUATE RESEARCH THESIS

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ABSTRACT

This thesis focuses on the fundamental theory of locally solid Riesz spaces. The interaction between the order and the topological structures of Riesz spaces will be demonstrated in this thesis. Before studying topological convergence, ideals and solid sets, we first describe a locally solid topology in terms of the uniform continuity of the lattice operations. Then, we concentrate on locally convex—solid Riesz. Finally, we look into the topological completion of a Hausdorff locally solid Riesz space.

LOCALLY SOLID TOPOLOGIES

DEFINITION : A linear topology τ on a vector space E that makes the addition and scalar multiplication continuous; that is, the topology τ makes the two functions

(x, y) → x+y from E × E to E

(λ, y) → λx from ℝ × E to E

continuous. A vector space E with a linear topology τ is known as a topological vector space (E, τ).

DEFINITION (Convex) : A subset A of E is called convex if $\lambda u + (1-\lambda)v \in A$ holds for all $u, v \in A$ and all $0 \leq \lambda \leq 1$.

DEFINITION (Locally Convex) : A locally convex topology on a vector space is a linear topology that has a base at zero consisting of convex sets. A linear topology τ on a vector space E is locally convex if and only if there is a family $\{\rho_\alpha\}_{\alpha \in A}$ of seminorms on E that generates τ.

DEFINITION (Seminorm) : A seminorm p on a vector space E is a function ρ: E→ ℝ satisfy the following three properties.

(1) ρ(u) ≥ 0 for all u ∈ E.

(2) ρ(u+v) ≤ ρ(u)+ ρ(v) for all u,v ∈ E.

(3) ρ(λu)= $|\lambda|\rho$(u) for all λ ∈ ℝ and u ∈ E.

DEFINITION (Topological Dual) : The topological dual E' of a topological vector space (E,τ) is the set of all τ−continuous linear functionals on E. In ordinary algebraic operations, E' is a vector space. E' is the vector subspace of the algebraic dual E*, the vector space of all linear functionals on E.

DEFINITION (Weak Topology) : The weak topology σ(E, E') of a locally convex space (E,τ) is the locally convex topology generated on the vector space E by the family of seminorms {ρ_f : f ∈ E'}, where ρ_f (u) = **|**f(u)**|** for u ∈ E.

DEFINITION (Strong Topology) : The strong topology β(E', E) on E' is the locally convex topology generated by the family of seminorms {ρ_A: A ∈ β}, where ρ_A(f) = sup{|f(u)|: u ∈ A} and β is the collection of all σ(E, E')−bounded subsets of E.

DEFINITION (Uniformly Continuous) : A function f: A → (E_1, τ_1), from a subset A of a topological vector space (E, τ) to another topological vector space (E_1, τ_1), is called uniformly continuous if for every τ_1−neighborhood V of zero of E_1 there exists a τ−neighborhood W of zero of E such that f(u) − f(v) ∈ V whenever u,v ∈ A satisfy u−v ∈ W.

DEFINITION (Topological Completion) : If (E,τ) is a Hausdorff topological vector space, then the unique (up to isomorphism) Hausdorff topological vector space (Ê, $\hat{\tau}$) described in Theorem 1.1.22 is called the topological completion of (E,τ).

DEFINITION (Quotient Topology) : For every mapping f from a topological space (X,τ) to a set Y, the finest topology τ_f on Y for which the function f is continuous is called the quotient topology determined by f. Then we have τ_f = {A ⊆ Y: f^{-1}(A) ∈ τ}.

DEFINITION (Polar - Bipolar) : For a subset A of a topological vector space (E, τ) the polar of A is defined by A° = {ϕ ∈ E' : |$\phi(u)$| ≤ 1 for all u ∈ A}.

The bipolar of a subset A of E is the set (A°)°, the polar of A°, and is denoted by A°°.

DEFINITION (Solid – Locally Solid) : A subset S of a Riesz space L is said to be solid if |u| ≤ |v| and v ∈ S imply u ∈ S.

A locally solid Riesz space (L, τ) is a Riesz space L equipped with a locally solid topology τ.

THEOREM (Roberts-Namioka) : For a linear topology τ on a Riesz space L the following statements are equivalent.

(1) (L, τ) is a locally solid Riesz space.

(2) The mapping (u, v) ↦ u ∨ v, from (L, τ)×(L, τ) to (L, τ), is uniformly continuous.

(3) The mapping (u, v) ↦ u ∧ v, from (L, τ)×(L, τ) to (L, τ), is uniformly continuous.

(4) The mapping u ↦ $|u|$, from (L, τ) to (L, τ), is uniformly continuous. .

(5) The mapping u ↦ u^-, from (L, τ) to (L, τ), is uniformly continuous.

(6) The mapping u ↦ u^+, from (L, τ) to (L, τ), is uniformly continuous.

DEFINITION (Locally Convex-Solid Topology) : A locally convex-solid topology is a linear topology τ on a Riesz space L that is both locally solid and locally convex.

DEFINITION (Locally Convex-Solid Riesz Space) : A locally convex−solid Riesz space (L,τ) is a Riesz space L equipped with a locally convex-solid topology τ.

DEFINITION (Riesz Seminorm – Riesz Norm) : A seminorm ρ on a Riesz space L is said to be a Riesz (or a lattice) seminorm if $|u| \leq |v|$ in L implies ρ(u) ≤ ρ(v), or equivalently, whenever ρ(u)=ρ(|u|) holds for all u ∈ L and 0 ≤ u ≤ v in L implies ρ(u) ≤ ρ(v).

A Riesz (or a lattice) norm is a Riesz seminorm that is also a norm.

THEOREM : Let L be a Riesz space and τ a linear topology on L. Then the following statements are equivalent.

i. τ is a locally convex−solid topology.

ii. There exists a family $\{\rho_\alpha\}_{\alpha \in A}$ of Riesz seminorms that generates the topology τ.

DEFINITION (Riesz Pseudonorm) : A Riesz pseudonorm ρ is a real−valued function defined on a Riesz space L (i.e., ρ : L → ℝ) satisfying the following properties.

(1)ρ(u) > 0 for each u ∈ L.

(2)ρ(u+v) ≤ ρ(u) + ρ(v) for all u, v ∈ L.

(3)ρ(λu) → 0 as λ → 0 for each u ∈ L.

(4)ρ(u) ≤ ρ(v) whenever |u| ≤ |v| holds in L.

DEFINITION : Let L be a Riesz space and let A be a nonempty subset of the order dual $L^\sim$. The absolute weak topology |σ|(L, A) generated by A on L is the locally convex−solid topology generated by the collection of Riesz seminorms {ρ_ϕ: ϕ ∈ A}, where ρ_ϕ(u)=|ϕ|(|u|).

DEFINITION (Monotone Completeness Property) : A locally solid Riesz space (L, τ) is said to satisfy the monotone completeness property (denoted by MCP), if every monotone τ−Cauchy net of L is τ−convergent in L.

DEFINITION (Dedekind Complete – Dedekind σ−Complete) : A Riesz space L is said to be:

(1) Dedekind (or order) complete, if every nonempty subset of L that is bounded from above has a supremum.

(2) Dedekind (or order) σ−complete, if every countable subset of L that is bounded from above has a supremum.

DEFINITION (Disjoint Complement) : The disjoint complement A^d of a nonempty subset A of a Riesz space L is defined by $A^d = \{u \in L : u \perp v \; for \; all \; v \in A\}$.

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